# The Gordon-Haus effect for modified NLS solitons

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**Abstract**

Random jitter in the soliton arrival time (the Gordon-Haus effect) is analyzed for solitons being solutions of the integrable modified nonlinear Schrödinger equation. It is shown that the mean square fluctuation of the soliton position depends on the soliton parameters which can be properly adjusted to suppress the Gordon-Haus jitter.

## 1 Introduction

Long-distance soliton-based fibre transmission meets with difficulties imposed by random jitter in the soliton arrival time caused by the spontaneous amplifier noise (the so-called Gordon-Haus effect [1]).The amplifier noise incorporated by the soliton produces a random soliton frequency shift which leads to a timing shift because of the group velocity dispersion. The Gordon-Haus effect limits error-free propagation of the soliton. There is a few ways to partially overcome the Gordon-Haus limit, including the use of linear filtering [2, 3], the dispersion compensation means [4, 5, 6], the use of a sequence of two different media to reduce a path-averaged dispersion [7, 8].

From the viewpoint of the inverse scattering transform, the theory of the Gordon-Haus effect is a direct consequence of the adiabatic soliton perturbation theory for the nonlinear Schrödinger (NLS) equation, with perturbation being the amplifier noise [9]. The NLS equation serves as the integrable model describing the picosecond soliton dynamics in fibres. On the other hand, when dealing with ultrashort optical pulses with duration $\leq 100$ fs, the NLS equation should be modified to adopt more subtle effects, such as the nonlinearity dispersion, the Raman self-frequency shift and the third-order dispersion [10]. It is remarkable that the account for the nonlinearity dispersion does not break the integrability of the equation. In other words, the modified NLS (MNLS) equation

$$iu_z + \frac{1}{2}u_{tt} + |u|^2 u + i\alpha(|u|^2 u)_t = 0. \tag{1}$$

is still integrable by means of the inverse scattering transform [11], though the linear spectral problem associated with the MNLS equation differs from that for the NLS equation (with $\alpha = 0$). Here $u$ is the normalized slowly varying amplitude of the electric field envelope, $z$ and $t$ are the normalized propagation distance and time in the frame comoving with the group velocity, the real parameter $\alpha$ governs the effect of the nonlinearity dispersion. We consider the MNLS equation (1) as the integrable model for ultrashort optical pulses, i.e., playing the same role as the NLS equation does for picosecond solitons. Thereby, we change the status of the nonlinearity dispersion term from being a perturbation in the NLS equation to the essential ingredient of the MNLS equation. It is important that such a change is in no way an issue of our convenience. It was shown by Ohkuma *et al.* [12] that numerical simulation of soliton propagation revealed a number of features which cannot be accounted for by treating this term as a perturbation term in the NLS equation. In the recent paper [13], the adiabatic perturbation theory for MNLS solitons was elaborated. So, a question to analyze the Gordon-Haus effect for the MNLS solitons arises naturally.

In this communication, we derive analytically the mean-square displacement fluctuation for the MNLS soliton propagating in a fibre. It follows from our results that the Gordon-Haus effect for the MNLS solitons, as distinct from the NLS solitons, can be suppressed by properly adjusting the soliton/fibre parameters, without making use of external means. In the end of the paper we briefly discuss the suitability of this result to actual femtosecond solitons.

## 2 Formulation of the problem

We consider the perturbed MNLS equation

$$iu_z + \frac{1}{2}u_{tt} + |u|^2 u + i\alpha(|u|^2 u)_t = s(z,t), \tag{2}$$

where $s(z,t)$ stands for spontaneous amplifier noise. In order to digress the details of minor importance, we consider distributed gain that exactly compensates for the fibre loss. Further, we consider the amplifier bandwidth to be much larger than the soliton bandwidth. Because erbium-doped amplifiers have a gain bandwidth of the order of 40 nm [9], this assumption works well for the ultrashort pulses. We consider the noise $s(z,t)$ as the delta-correlated function both in time and space,

$$< \bar{s}(z,t)s(z',t') >= A\delta(t-t')\delta(z-z'). \tag{3}$$

The estimation of the coefficient $A$ can be found in [9]. Finally, we treat the noise $s(z,t)$ as being small to justify the perturbative approach.

The unperturbed soliton of the MNLS equation (1) was derived as early as in 1983 by Gerdjikov and Ivanov [14]. We will use more simple and transparent expression [13] for the soliton solution of (1) :

$$u_s(z,t) = \frac{i}{w}\frac{ke^{-x} + \bar{k}e^x}{(ke^x + \bar{k}e^{-x})^2}e^{i\psi}. \tag{4}$$

Here $x$ and $\psi$ are linearly expressed through coordinates $z$ and $t$ :

$$x = -\frac{t}{w} + q(z), \qquad \psi = vt + \phi(z), \qquad q(z) = a + \frac{v}{w}z, \qquad \phi(z) = \varphi - \frac{1}{2}(v^2 - \frac{1}{w^2})z. \tag{5}$$

Parameters $a$ and $\varphi$ determine initial position and initial phase of the soliton, the complex parameter $k = \xi - i\eta$, $\xi, \eta > 0$ determines velocity (more exactly, shift of the reciprocal soliton velocity) $v$ and width $w$ of the soliton,

$$v(z) = \frac{1}{\alpha} - \frac{2}{\alpha}\left(k^2 + \bar{k}^2\right), \qquad w(z) = -\frac{i\alpha}{2(k^2 - \bar{k}^2)}, \tag{6}$$

where the $z$-dependence of the soliton velocity and width arises from the perturbation-induced $z$-dependence of the parameter $k$ (see below (8)). Functions $q(z)$ and $\phi(z)$ are the direct analogs of the Gordon's position and phase parameters for the NLS soliton [15].

The soliton (4) has a number of peculiarities which distinguish it from the NLS soliton [13]. For example, the important invariant of eq. (1), namely, the optical energy

$$E = \int\limits_{-\infty}^{\infty} dt|u|^2 = \frac{4}{\alpha}\gamma \qquad \gamma = \text{Arg}(\bar{k}), \qquad 0 < \gamma < \pi/2,$$

2

has the upper limit $2\pi/\alpha$. It should be stressed that $\alpha$ enters the denominator of the soliton solution (4) (because of $w$), so we account non-perturbatively for the nonlinearity dispersion. In other words, the parameter $\alpha$ does not to be small in general. Nevertheless, a non-trivial limiting procedure exists [13] permitting to restore from (4) the NLS soliton as $\alpha \to 0$. Indeed, let us take the limit

$$k = \frac{1}{2} - \frac{\alpha}{2}k_{\text{NLS}} + \mathcal{O}(\alpha^2), \qquad k_{\text{NLS}} = \xi_{\text{NLS}} + i\eta_{\text{NLS}}. \tag{7}$$

Then it is easy to see that the soliton velocity and width (6) transform in the limit (7) as $v \to 2\xi_{\text{NLS}}$, $w \to 1/2\eta_{\text{NLS}}$, i. e., to the NLS parameters, while the MNLS soliton (4) produces exactly the NLS soliton $u_{\text{NLS}} = 2i\eta_{\text{NLS}}\exp(i\psi_{\text{NLS}})\text{sech}x_{\text{NLS}}$ with $x_{\text{NLS}} = a - 2\eta_{\text{NLS}}(\tau - 2\xi_{\text{NLS}}z)$, $\psi_{\text{NLS}} = \phi + 2\xi_{\text{NLS}}\tau - 2(\xi_{\text{NLS}}^2 - \eta_{\text{NLS}}^2)z$.

As it was shown in [13], a perturbation-induced z-evolution of the parameter $k$ is given by the simple formula:

$$\frac{dk}{dz} = \frac{i}{2}\alpha k^2 \int_{-\infty}^{\infty} dx \frac{e^x}{(ke^{-x} + \bar{k}e^x)^2}[s(x)e^{-i\psi} + \bar{s}(-x)e^{i\psi}]. \tag{8}$$

which is transformed in accordance with (6) into the corresponding relations for velocity and width:

$$\frac{dv}{dz} = -2i \int_{-\infty}^{\infty} dx \frac{k^3 e^x - \bar{k}^3 e^{-x}}{(ke^{-x} + \bar{k}e^x)^2}[s(x)e^{-i\psi} + \bar{s}(-x)e^{i\psi}] \equiv S_v(z), \tag{9}$$

$$\frac{d}{dz}\frac{1}{w} = -2 \int_{-\infty}^{\infty} dx \frac{k^3 e^x + \bar{k}^3 e^{-x}}{(ke^{-x} + \bar{k}e^x)^2}[s(x)e^{-i\psi} + \bar{s}(-x)e^{i\psi}] \equiv S_w(z).$$

Here $S_v(z)$ and $S_w(z)$ stand for the noise sources driving velocity and width. Because noise $s(z,t)$ is given in terms of the correlation function (3), the responses are also expressed in the same form.

We are mostly interested in the mean-square fluctuation $\langle q(L)q(L)\rangle$ ($L$ being a fibre length) of the soliton displacement $q(z)$ (5) which is given in the presence of perturbation by

$$q(z) = a + \int^{z} dz' v(z')w^{-1}(z').$$

with $v$ and $w$ determined from (9). The jitter of the soliton displacement in the comoving frame is calculated from

$$\frac{dq}{dz} = \frac{da}{dz} + \frac{v}{w}, \tag{10}$$

where z-evolution of $a$ is given by [13]

$$\frac{da}{dz} = \frac{da_+}{dz} + \frac{da_-}{dz} \equiv S_a(z), \quad \frac{da_+}{dz} = wq(z)S_w(z), \tag{11}$$

$$\frac{da_-}{dz} = \int_{-\infty}^{\infty} \frac{dx}{(ke^{-x} + \bar{k}e^x)^2}\left[\frac{i\alpha}{2}(ke^x + \bar{k}e^{-x}) + 4wx(k^3 e^x + \bar{k}^3 e^{-x})\right][s(x)e^{-i\psi} - \bar{s}(-x)e^{i\psi}].$$

3

Since the soliton velocity and width enter eq. (10) in the combination $vw^{-1}$, we find from (9) the noise source $S_{vw^{-1}}$:

$$
\begin{aligned}
\frac{d}{dz}(\frac{v}{w}) &= -2k^3(v + \frac{i}{w}) \int\limits_{-\infty}^{\infty} dx \frac{e^x}{(ke^{-x} + \bar{k}e^x)^2} [s(x)e^{-i\psi} + \bar{s}(-x)e^{i\psi}] \\
&- 2\bar{k}^3(v - \frac{i}{w}) \int\limits_{-\infty}^{\infty} dx \frac{e^{-x}}{(ke^{-x} + \bar{k}e^x)^2} [s(x)e^{-i\psi} + \bar{s}(-x)e^{i\psi}] \equiv S_{vw^{-1}}(z).
\end{aligned}
$$
(12)

Finally, note that the mean-square fluctuation of the quantity $B$ caused by the noise source $S_B$ is determined by

$$
\langle \bar{B}(z)B(z') \rangle = \langle \int\limits_0^z d\zeta \bar{S}_B(\zeta) \int\limits_0^{z'} d\zeta' S_B(\zeta') \rangle.
$$

Now we have everything to calculate the mean-square fluctuation of the soliton displacement.

## 3  Results

It is seen from (10) that the mean-square fluctuation of the soliton displacement is produced by both the noise source $S_a$ driving the displacement directly and the fluctuations of the combination $vw^{-1}$. We consider noise sources as independent, so the fluctuations they produce are additive. This gives

$$
\langle q(L)q(L) \rangle = \langle q(L)q(L) \rangle_a + \langle q(L)q(L) \rangle_{vw^{-1}}.
$$

Here

$$
\langle q(L)q(L) \rangle_{vw^{-1}} = \int\limits_0^L dz \int\limits_0^L dz' \langle \frac{v}{w}(z) \frac{v}{w}(z') \rangle.
$$
(13)

and

$$
\langle q(L)q(L) \rangle_a = \int\limits_0^L dz \int\limits_0^L dz' \langle \bar{S}_a(z)S_a(z') \rangle.
$$

In turn, the correlation function for the velocity-to-width ratio is determined by the noise source $S_{vw^{-1}}$(12):

$$
\langle \frac{v}{w}(z) \frac{v}{w}(z') \rangle = \int\limits_0^z d\zeta \int\limits_0^{z'} d\zeta' \langle \bar{S}_{vw^{-1}}(\zeta)S_{vw^{-1}}(\zeta') \rangle.
$$
(14)

Straightforward calculation by means of eqs. (3) and (12) gives

$$
\langle \bar{S}_{vw^{-1}}(z)S_{vw^{-1}}(z') \rangle = AF(\alpha, v, \gamma)\delta(z - z')
$$
(15)

with

$$
\begin{aligned}
F(\alpha, v, \gamma) &= \frac{4}{\alpha^2}(1 - \alpha v)\{(1 - 2\alpha v)[(1 - \alpha v)\sec^2 2\gamma - 2(1 - 2\alpha v)] \\
&+ ([(1 - \alpha v)\sec^2 2\gamma - 2(1 - 2\alpha v)]^2 + (1 - 2\alpha v)^2 \sec^2 2\gamma)\gamma \cot 2\gamma\}.
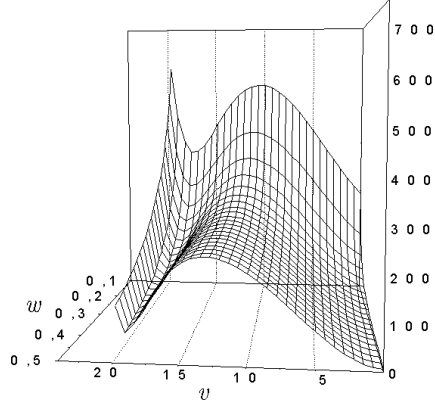\end{aligned}
$$
(16)

4

Figure 1: Profile of the function $F(\alpha, v, \gamma)$ determining the correlation function for the soliton velocity-to-width ratio for $\alpha = 0.05$. The soliton width $w$ is related with the parameter $\gamma$ through (17).

In the course of calculation, we ignore, because of smallness of noise sources, small variations of the parameter $k$. Thereby, the parameters $V, w$ and $\gamma$ refer to their initial values. (Remind that distributed gain compensates exactly for fiber losses). Besides, we express the soliton width $w$ in terms of $v$ and $\gamma$ due to the relation

$$w(1 - \alpha v) = \alpha \cot 2\gamma. \tag{17}$$

Finally, we obtain from (14) and (13) the correlation function for the soliton displacement driven by fluctuations of the velocity-to-width ratio:

$$\langle q(L)q(L)\rangle_{vw^{-1}} = \frac{1}{3}AF(\alpha, v, \gamma)L^3. \tag{18}$$

Similar calculation for the $S_a$-driven correlation $\langle q(L)q(L)\rangle_a$ in accordance with (11) displays the linear growth with $L$. Hence, for large propagation distances the correlation function (18) provides the main contribution to the total mean-square fluctuation of the soliton displacement.

## 4 Discussion

The above dependences on $L$ of the mean-square MNLS soliton fluctuations coincide completely with those in the case of the NLS soliton [1, 9]. Nevertheless, there is a substantial difference between NLS and MNLS equations. While the correlation functions for the NLS soliton do not contain soliton parameters (in dimensionless units), these parameters enter explicitly into the r.h.s. of (15) for the MNLS equation. Therefore, we can reduce the Gordon-Haus jitter for the MNLS soliton by varying soliton parameters. In Fig. 1 we demonstrate this effect in terms of variables $v$ and $w$. It is seen that the function (16) smoothly decreases when $v$ approaches $\alpha^{-1}$.

The role of the optical energy $\gamma$ is depicted on Fig. 2. As follows from (17), we have $0 < \gamma < \pi/4$ for $v < \alpha^{-1}$ and $\pi/4 < \gamma < \pi/2$ for $v > \alpha^{-1}$. It is seen that the function (16) grows significantly for all $\gamma$ for $v > \alpha^{-1}$, while the same growth for $v < \alpha^{-1}$ is displayed to a lesser extent, including non-monotone behavior for small $\gamma$.
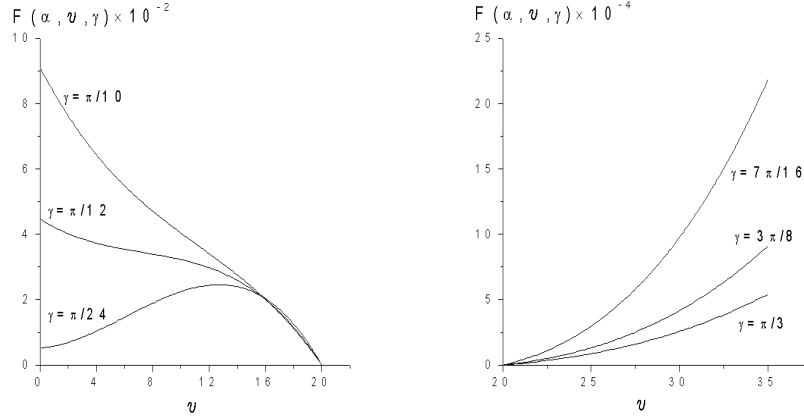
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Figure 2: Typical behavior of the function $F(\alpha, v, \gamma)$ versus $v$ for different $\gamma$ and $\alpha = 0.05$; (a) $v < \alpha^{-1}$, (b) $v > \alpha^{-1}$.

For slightly different definition of the soliton position as $x = -\frac{1}{w}(t - q(z))$ the function $F(\alpha, v, \gamma)$ takes more simple form,

$$F(\alpha, v, \gamma) = 2(1 - \alpha v)\{1 - \cot^2 2\gamma + \gamma[(1 - \tan^2 2\gamma)^2 + \sec^2 2\gamma]\cot^3 2\gamma\},$$

but with the same qualitative behavior.

As regards the applicability of the above result on the soliton jitter suppression to actual femtosecond optical solitons, we believe that the MNLS equation is the true integrable model to start with the account for the third-order dispersion and the Raman self-frequency shift. It should be noted that in the adiabatic approximation the third-order dispersion does not contribute to the MNLS soliton velocity and width, just as for the NLS soliton [16]. The Raman effect contribution can be calculated additively as an extra perturbation to the MNLS soliton, within the framework of the MNLS soliton perturbation theory [13]. In so doing, a value of $\alpha$ should be carefully reconciled with the specific conditions of femtosecond pulse propagation. It is worth noting that there exists a possibility [17] to partially compensate for the Raman self-frequency shift effect, as applied to femtosecond optical pulses. Corresponding results will be published elsewhere.

In conclusion, we have shown that for solitons of the MNLS equation which can serve as the integrable model for the description of ultrashort soliton dynamics, the Gordon-Haus jitter can be significantly reduced by means of matching soliton parameters alone. We stress that this reduction is non-perturbative with respect to $\alpha$, i.e., it cannot be revealed in the framework of the NLS equation with $\alpha$-dependent term considered as a perturbation. Though the above result is valid within the adiabatic approximation of the MNLS soliton perturbation theory, we believe it describes the main features of this phenomenon for ultrashort optical pulses.

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